Math and Coding II: Term Exam (114–2)

80 minutes, full mark = 35

It is **strictly forbidden** to use your notebooks/memos/books except for one tiny "cheat sheet".
It is **strictly forbidden** to **have/wear/use** any electric devices, even in your pockets.
It is **strictly forbidden** to discuss with other students.

Administrative Remarks

- Use full-sized sheets for your answers. Smaller sheets are for calculation, not to submit.
- Write your name and student ID on the answer sheet. Put your student ID on the desk.
- Allowed on desk: student ID card (required), pens/pencils, correction tools (eraser etc.), rulers, a "cheat sheet" not larger than student ID, and drinks. **Others must be stored in your bags.**
- You cannot wear watches nor electronic devices. **You cannot have them even in your pockets**.
- **After 13:10, the following actions are considered cheating. You may immediately lose your credit.**
  - If non-allowed items (pen cases, foods, poaches, etc.) are found on desks.
  - If you have textbooks, mobile phones, tablets, or PC, if they are not stored in your bags, or if you use them. They must be in your bags even after you submit your answer sheets.
- Breaks are not allowed in principle. After 14:00, you may leave after submission. In case of health problems or other issues, call the TA or lecturer.
- *Any form of academic dishonesty, including chats, additions/corrections after the period, and using your phones, will be treated by NSYSU "Academic Regulations."*

Scientific Remarks

- Show your calculations or thought process for **partial mark**!
- Use English, where mistakes are tolerated. Meanwhile, scientific mistakes are not tolerated.
- If you find any errors or issues in the questions, explain them on your answer sheet, make necessary adjustments on the question, and answer accordingly.
- You may use the following notations and values without definition/declaration.

$\lvert A\rvert$	determinant of a matrix A (equivalent to $\det A$).
$\lVert\vec{v}\rVert$ or $\lvert\vec{v}\rvert$	norm (magnitude) of a vector $\vec{v}$.
I_n or I	$n\times n$ identity matrix, possibly with n understood.
$O_{m,n}$ or O	$m\times n$ zero matrix, possibly with m and n understood.
$\overline{A}$	complex conjugate of a matrix A (or a complex number).
A^{T}	transpose of A.
$A^{\dagger}$	Hermitian conjugate of A.
$\operatorname{Re} z$, $\operatorname{Im} z$	real/imaginary part of a complex number z.
$\mathbb{R}$, $\mathbb{C}$	the set of all real/complex numbers.
$\mathbb{R}^n$, $\mathbb{C}^n$	the set of all n-dimensional real/complex vectors.
$M^{m,n}$	the set of all $m\times n$ real matrices.
$M^{m,n}(\mathbb{C})$	the set of all $m\times n$ complex matrices.

$\sqrt{2}\approx 1.414$ $\sqrt{3}\approx 1.732$ $\sqrt{5}\approx 2.236$ $\sqrt{7}\approx 2.646$ $\pi\approx 3.142$ $\mathrm{e}\approx 2.718$

Further formulae and definitions are found in the other side of this exam sheet.

Answer **[Part A]**–**[Part D]**. If you still have time, answer **[Part E]**.

[Part A] Numerical Analysis Fundamentals (7 points)

Sho runs the following Python code (the left panel) and obtains the output shown in the right panel.

```
1  a = pow(10.0, 100)
2  print(a)
3  print(a * a * a)
4  print(a * a * a * a)
5  print("------")
6  b = pow(10.0, 10)
7  print(a + b == a)
8  print("------")
9  print(0.4 + 0.8 == 1.2)
10 print(0.25 + 4.5)
```

```
1  1e+100
2  1e+300
3  inf
4  ------
5  True
6  ------
7  False
8  4.75
```

This strange result is due to the following properties of the `float` data type:

Ⓐ We use binary numbers, $m \times 2^p$, to express real numbers.
Ⓑ The exponent p is limited to 13 bits,
Ⓒ The mantissa m is limited to 52 bits,

Now, let's focus on the output $\boxed{\texttt{inf}}$ in Line 3. It should be $a^4 = 10^{400}$, but a different output is given.

(1) This is caused by one of the above Ⓐ–Ⓒ. Choose the most relevant one from Ⓐ–Ⓒ.

(2) This behavior happens for $a^4 = 10^{400}$, while $a^3 = 10^{300}$ is correctly displayed. Explain the reason quantitatively. You may use $\log_{10} 2 \approx 0.3010$.

Next, consider the output $\boxed{\texttt{True}}$ in Line 5. Here, $b = 10^{10}$ but it says $a + b$ is equal to a.

(3) This is caused by one of the above Ⓐ–Ⓒ. Choose the most relevant one from Ⓐ–Ⓒ.

(4) If $b = 10^{80}$, what will be the output? How about $b = 10^{90}$? Guess the result with reason.

Finally, see the last two lines of the code.

(5) Express 0.25, 4.5, and 4.75 in binary. [Hint: For example, five (5) is expressed as 101 in binary.]

(6) Why is the output (Line 7) $\boxed{\texttt{False}}$? Give a brief explanation.

[Part B] Gamma function (11 points)

(1) Calculate $\Gamma(1)$. [Hint: Show the process. (Also in the following problems.)]

(2) Gamma function satisfies $x\,\Gamma(x) = \Gamma(x+1)$ for any $x > 0$. Prove it, using integration-by-parts.

(3) Calculate $\Gamma(7)$.

(4) Calculate $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$. [Hint: The answer is $\sqrt{\pi}$.]

(5) Calculate $\Gamma(1/2)$ and $\Gamma(5/2)$.

[The exam questions continue on the next page.]

[Part C] Fourier analysis (13 points)

(1) Calculate $\displaystyle\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right) \mathrm{d}x$, where $L > 0$ and $\underline{n \text{ is an integer}}$.
[Hint: Show the process. (Also in the following problems.)]

(2) Calculate $\displaystyle\int_{-\pi}^{\pi} x\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}x$, where k is an integer.

Now, consider $f(x) = |x|$ defined for $-\pi \le x \le \pi$ and extended periodically with period 2π, so that we can analyze it by Fourier series with the period $2L = 2\pi$, i.e.,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$

(3) Find a_0, a_n, and b_n, where n is a positive integer.

[Part D] Special functions (4 points)

In this part, you are asked to find a non-trivial solution, $y(x) \neq 0$, of differential equations. You can write the answer in terms of special functions, i.e., you do not have to find the explicit form of the function or the polynomial. Just write the answer, and no need to provide reasoning or derivation.

(1) Find one non-trivial solution of $y'' - 2xy' + 2ny = 0$, where n is a non-negative integer.

(2) Find one non-trivial solution of $x^2y'' + xy' + (x^2 - 1)y = 0$.

[Part E] Extra Problems

(1) Find the general solution of $x^2y'' + xy' + (x^2 - 1)y = 0$.

(2) Find Fourier series expansion of $f(x) = |\sin x|$.

(3) Find an ordinary different equation whose solution is given by $j_n(x)$, where $j_n(x)$ is the spherical Bessel function with order n, with n being a non-negative integer.

[This is the end of the exam questions.]

Fourier series for periodic functions $f(x)$ with period $2L$ is defined by

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right), \quad \text{where } L > 0,\ n = 1, 2, 3, \ldots, \text{ and}$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,\mathrm{d}x, \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,\mathrm{d}x, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\,\mathrm{d}x.$$

Fourier inversion theorem states

$$f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \right] \mathrm{e}^{\mathrm{i}ky}\,\mathrm{d}k$$

for many functions, e.g., $f(x)$ such that $f(x)$, $f'(x)$ and $f''(x)$ exists and are integrable over $\mathbb{R}$, or $f(x)$ being finite and continuous for any $x \in \mathbb{R}$ and with $\int_{-\infty}^{\infty} |f(x)| \mathrm{d}x$ being finite.

Normal distribution with mean $\mu \in \mathbb{R}$ and standard deviation $\sigma > 0$ is given by

$$N_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2} \right].$$

Special functions are defined as follows.

Gamma function: $\Gamma(z) = \int_0^{\infty} t^{z-1} \mathrm{e}^{-t}\,\mathrm{d}t \quad \text{for} \quad z \in \mathbb{C}, \quad \operatorname{Re} z > 0.$

Beta function: $\mathrm{B}(x, y) = \dfrac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \quad \text{for} \quad x, y \in \mathbb{C}, \quad \operatorname{Re} x > 0, \quad \operatorname{Re} y > 0.$

Legendre polynomials: $P_n(x) = \dfrac{1}{2^n n!} \dfrac{\mathrm{d}^n}{\mathrm{d}x^n} \left[(x^2 - 1)^n \right] \quad \text{for} \quad n = 0, 1, 2, \ldots.$

It is a solution of Legendre's differential equation $(1 - x^2) y'' - 2xy' + n(n+1) y = 0.$

Laguerre polynomials: $L_n(x) = \dfrac{\mathrm{e}^x}{n!} \dfrac{\mathrm{d}^n}{\mathrm{d}x^n} (\mathrm{e}^{-x} x^n) \quad \text{for} \quad n = 0, 1, 2, \ldots.$

It is a solution of Laguerre's differential equation $xy'' + (1 - x) y' + ny = 0.$

Hermite polynomials: $H_n(x) = (-1)^n \mathrm{e}^{x^2} \dfrac{\mathrm{d}^n}{\mathrm{d}x^n} \left(\mathrm{e}^{-x^2} \right) \quad \text{for} \quad n = 0, 1, 2, \ldots.$

It is a solution of Hermite's differential equation $y'' - 2xy' + 2ny = 0.$

Bessel functions of the first kind: $J_\nu(x) = x^\nu \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\ \Gamma(\nu + m + 1)} \quad \text{for} \quad \nu \in \mathbb{C}.$

Bessel functions of the second kind: $Y_\nu(x) = \dfrac{J_\nu(x) \cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \quad \text{for non-integer } \nu,$

$$Y_\nu(x) = \lim_{\mu \to \nu} Y_\mu(x) \quad \text{for integer } \nu.$$

$J_\nu(x)$ and $Y_\nu(x)$ are two linearly independent solutions of Bessel's differential equation $x^2 y'' + xy' + (x^2 - \nu^2) y = 0.$

Other Bessel functions: $I_\nu(x) = \mathrm{i}^{-\nu} J_\nu(\mathrm{i}x), \quad K_\nu(x) = \dfrac{\pi}{2\sin(\nu\pi)} \left[I_{-\nu}(x) - I_\nu(x) \right],$

$$j_\nu(x) = \sqrt{\frac{\pi}{2x}}\, J_{\nu+1/2}(x), \quad y_\nu(x) = \sqrt{\frac{\pi}{2x}}\, Y_{\nu+1/2}(x).$$

$j_\nu(x)$ and $y_\nu(x)$ are called spherical Bessel functions.