# Math and Coding II: Midterm Exam (114–2) **100 minutes, full mark = 45**

It is **strictly forbidden** to use your notebooks/memos/books.
It is **strictly forbidden** to **have/wear/use** any electric devices, even in your pockets.
It is **strictly forbidden** to discuss with other students.

### Administrative Remarks

- Use full-sized sheets for your answers. Smaller sheets are for calculation, not to submit.
- Write your name and student ID on the answer sheet. Put your student ID on the desk.
- Allowed on your desk: student ID card (required), pens/pencils, correction tools (eraser etc.), rulers, and drinks. **Other items must be stored in your bags.**
- You cannot wear watches nor electronic devices. **You cannot have them even in your pockets**.
- **After 13:10, the following actions are considered cheating. You may immediately lose your credit.**
  - If non-allowed items (pen cases, foods, poaches, etc.) are found on desks.
  - If you have textbooks, mobile phones, tablets, or PC, if they are not stored in your bags, or if you use them. They must be in your bags even after you submit your answer sheets.
- Breaks are not allowed in principle. After 14:00, you may leave after submission. In case of health problems or other issues, call the TA or lecturer.
- *Any form of academic dishonesty, including chats, additions/corrections after the period, and using your phones, will be treated by NSYSU "Academic Regulations."*

### Scientific Remarks

- Show your calculations or thought process for **partial mark**!
- Use English, where mistakes are tolerated. Meanwhile, scientific mistakes are not tolerated.
- If you find any errors or issues in the questions, explain them on your answer sheet, make necessary adjustments on the question, and answer accordingly.
- You may use the following notations and values without definition/declaration.

| | |
|---|---|
| $\lvert A\rvert$ | determinant of a matrix $A$ (equivalent to $\det A$). |
| $\lVert\vec{v}\rVert$ or $\lvert\vec{v}\rvert$ | norm (magnitude) of a vector $\vec{v}$. |
| $I_n$ or $I$ | $n\times n$ identity matrix, possibly with $n$ understood. |
| $O_{m,n}$ or $O$ | $m\times n$ zero matrix, possibly with $m$ and $n$ understood. |
| $\overline{A}$ | complex conjugate of a matrix $A$ (or a complex number). |
| $A^{\mathrm{T}}$ | transpose of $A$. |
| $A^{\dagger}$ | Hermitian conjugate of $A$. |
| $\operatorname{Re} z$, $\operatorname{Im} z$ | real/imaginary part of a complex number $z$. |
| $\mathbb{R}$, $\mathbb{C}$ | the set of all real/complex numbers. |
| $\mathbb{R}^n$, $\mathbb{C}^n$ | the set of all $n$-dimensional real/complex vectors. |
| $M^{m,n}$ | the set of all $m\times n$ real matrices. |
| $M^{m,n}(\mathbb{C})$ | the set of all $m\times n$ complex matrices. |

$\sqrt{2}\approx 1.414$ $\sqrt{3}\approx 1.732$ $\sqrt{5}\approx 2.236$ $\sqrt{7}\approx 2.646$ $\pi\approx 3.142$ $\mathrm{e}\approx 2.718$

Answer **[Part I]**–**[Part IV]**. If you still have time, answer **[Part V]**.
However, you will NOT get the point from **[Part V]** if your score for **[Part I]**–**[Part IV]** is less than 70%.

## [Part I] Matrix Operations (28 points)

Consider the following numbers, vectors, and matrices. Here, $\theta$ is a real number.

$$\alpha = 4 + 3\mathrm{i} \qquad \beta = 2 - \mathrm{i}$$

$$\vec{u} = \begin{pmatrix} 0 \\ 1+\mathrm{i} \\ 1-\mathrm{i} \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 1 \\ -2\mathrm{i} \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 1 \\ 1+\mathrm{i} \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 0 \\ 2 & 6 \\ 0 & 4 \\ 1 & 3 \end{pmatrix} \qquad D = \begin{pmatrix} 2\mathrm{i} & 0 \\ 1-\mathrm{i} & 3\mathrm{i} \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 2\mathrm{i} \\ 2\mathrm{i} & 0 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & -6\mathrm{i} \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

**(A)** [For this problem, only the final answer is required. No need to show detailed calculations.]
Calculate the following expressions. If it is not defined or it does not exist, answer so.
The definition of the inner product $\langle \vec{a} | \vec{b} \rangle$ is the same as $\vec{a} \cdot \vec{b}$ of the textbook, Section 8.5.

| | | | |
|---|---|---|---|
| **1)** $\alpha\overline{\beta}$ | **6)** $A^2$ | **11)** $AB$ | **16)** $\det P$ |
| **2)** $\lvert\alpha^2\rvert$ | **7)** $\operatorname{rank} A$ | **12)** $BC$ | **17)** $\operatorname{rank} P$ |
| **3)** $\lvert\vec{u}\rvert$ | **8)** $A^{-1}$ | **13)** $CB$ | **18)** $\operatorname{rank} Q$ |
| **4)** $\langle \vec{v} \vert \vec{w} \rangle$ | **9)** $\det B$ | **14)** $\operatorname{rank} C$ | **19)** $\det R$ |
| **5)** $\langle \vec{v} \vert S \vert \vec{v} \rangle$ | **10)** $B^{-1}$ | **15)** $D^\dagger$ | **20)** $\operatorname{rank} Z$ |

**(B)** Calculate **(1)** $\det(P^2)$ and **(2)** $\operatorname{rank} R$, showing your calculation or thought processes.

**(C)** [For this problem, only the final answer is required, in symbols $A$, $B$, ..., and/or $Z$.]
From the above matrices $A$, $B$, ..., $Z$, choose all the matrices that are

**1)** in a row echelon form.
**2)** upper triangular.
**3)** Hermitian.
**4)** unitary.

## [Part II] Linear System of Equations (2.5 points)

Consider the linear system of equations, $A\vec{x} = \vec{b}$, where $A = \begin{pmatrix} 4 & 1 & 0 \\ 5 & -3 & 1 \\ -9 & 2 & -1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$.

**(A)** Write down the augmented matrix $\widetilde{A}$.

**(B)** Find a row echelon form of $\widetilde{A}$ through elementary row operations, showing each step.

*[The exam questions continue on the next page.]*

## [Part III] Eigenvalues and Diagonalizaton (7.5 points)

Consider $A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 9 & -1 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

**(A)** Find eigenvalues and corresponding eigenvectors of $A$.

**(B)** Find a matrix $P$ such that $P^{-1}BP$ is diagonal.

**(C)** Find a unitary matrix $U$ such that $U^{-1}CU$ is diagonal.

## [Part IV] Properties of Matrices (12 points)

Prove the following statements.

**(A)** Hermitian matrices are normal.

**(B)** If $A^2 = A$, then $A$ is a square matrix. If, furthermore, $A$ is not an identity matrix, $A$ is singular.

**(C)** Eigenvalues $\lambda_i$ of unitary matrices satisfy $|\lambda_i| = 1$.

**(D)** If a matrix $A$ is Hermitian and $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors of $A$ corresponding to different eigenvalues, then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

## [Part V] Extra Problem

> This extra problem is for students who have done the previous parts quickly and have some spare time. You **do not** get any points for this problem if your score for **[Part I]–[Part IV]** is less than 70%.

**(A)** Calculate the following expressions, showing your thought process and calculations.[*1]

**(1)** $\begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}^{2026}$

**(2)** $\det \begin{pmatrix} 17 & 18 & 19 \\ 20 & 21 & 22 \\ 23 & 24 & -25 \end{pmatrix}$

**(B)** Prove the following statements.

- Any square matrix $M$ may be written as the sum of a Hermitian matrix $H$ and a skew-Hermitian matrix $S$. Furthermore, $M$ is normal if and only if $HS = SH$.

[*1]The first problem is based on the date today, April 13th, 2026. The second problem is taken from an advertisement of Furukawa Co., Ltd. （古河機械金属株式会社）displayed at subway stations in Tokyo in March 2026.

*[This is the end of the exam questions.]*