Math and Coding II: Midterm Exam (113–2)

100 minutes, full mark = 40

Use of your notebooks/memos/books:	Strictly forbidden.
Use of your mobile etc. & Internet:	Strictly forbidden.
Discussion with other attending students:	Strictly forbidden.

Administrative Remarks

- Write your name and student ID on the answer sheet. Put your student ID on the desk.
- Allowed on your desk: student ID card (required), pens/pencils, correction tools (eraser etc.), rulers, and drinks. **Other items must be stored in your bags.**
- You cannot wear watches nor electronic devices. **You cannot have them even in your pockets**.
- **After 13:10, the following actions are considered cheating. You may immediately lose your credit.**
  - If non-allowed items (pen cases, foods, poaches, etc.) are found on desks.
  - If you have textbooks, mobile phones, tablets, or PC, if they are not stored in your bags, or if you use them. They must be in your bags even after you submit your answer sheets.
- Breaks are not allowed in principle. After 14:00, you may leave after submission. In case of health problems or other issues, call the TA or lecturer.
- *Any form of academic dishonesty, including chats, additions/corrections after the period, and using your phones, will be treated by NSYSU "Academic Regulations."*

Scientific Remarks

- Show your calculations or thought process for **partial mark**!
- Use English, where mistakes are tolerated. Meanwhile, scientific mistakes are not tolerated.
- If you find any errors or issues in the questions, explain them on your answer sheet, make necessary adjustments on the question, and answer accordingly.
- You may use the following notations and values without definition/declaration.

$\lvert A\rvert$	determinant of a matrix A (equivalent to $\det A$).
$\lVert\vec{v}\rVert$ or $\lvert\vec{v}\rvert$	norm of a vector $\vec{v}$.
I_n	$n \times n$ identity matrix.
$O_{m,n}$	$m \times n$ zero matrix.
I	identity matrix with shape understood.
O	zero matrix with shape understood.
A^{T}	transpose of A.
$\overline{A}$	complex conjugate of A.
$A^\dagger$	Hermitian conjugate of A.
$\mathbb{R}, \mathbb{C}$	the set of all real/complex numbers.
$\mathbb{R}^n, \mathbb{C}^n$	the set of all n-dimensional real/complex vectors.
$M^{m,n}$	the set of all $m \times n$ real matrices.
$M^{m,n}(\mathbb{C})$	the set of all $m \times n$ complex matrices.

$\sqrt{2} \approx 1.414 \quad \sqrt{3} \approx 1.732 \quad \sqrt{5} \approx 2.236 \quad \sqrt{7} \approx 2.646 \quad \pi \approx 3.142 \quad \mathrm{e} \approx 2.718$

Answer **[Part I]**–**[Part IV]**. If you still have time, answer **[Part V]** and **[Part VI]**.
However, you will NOT get the point from **[Part VI]** if your score for **[Part I]**–**[Part IV]** is less than 75%.

[Part I] Matrix Operations (17.6 points)

Consider the following matrices and vectors, where a, b, x, and θ are real numbers:

$$A = \begin{pmatrix} 4 & -1 & 8 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{pmatrix} \qquad P = \begin{pmatrix} \cos a & -\sin a \\ \sin b & \cos b \end{pmatrix} \qquad \vec{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} \qquad Q = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 2\mathrm{i} \\ -2\mathrm{i} \end{pmatrix}$$

$$D = \begin{pmatrix} 1+\mathrm{i} & 0 \\ 1+\mathrm{i} & 0 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 1+\mathrm{i} \\ 1-\mathrm{i} \end{pmatrix}$$

(1) Calculate the following expressions. If it is not defined or it does not exist, answer so.

A)	$\operatorname{rank} A$	**E)**	$\operatorname{rank} C$	**I)**	$\det P$	**M)**	$\lvert\vec{v}\rvert$
B)	$\det A$	**F)**	$\det C$	**J)**	$\det(R^3)$	**N)**	$\vec{v}\cdot\vec{w}$
C)	$\det B$	**G)**	C^{-1}	**K)**	$S^\dagger S$		
D)	B^{-1}	**H)**	$D^\dagger$	**L)**	$\lvert\vec{a}\rvert$		

(2) Find the condition on x for the matrix Q to be an invertible matrix.

(3) Discuss $\operatorname{rank} Q$.

(4) Consider $\vec{c} = \begin{pmatrix} 3 \\ k \\ 1 \end{pmatrix}$. Describe the condition on k for $\{\vec{a}, \vec{b}, \vec{c}\}$ to be linearly **independent**.

[Part II] Linear System of Equations (6.4 points)

Consider the linear system of equations, $A\vec{x} = \vec{b}$, where $A = \begin{pmatrix} 0 & 4 & 3 \\ 2 & 0 & -1 \\ 3 & 2 & 0 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 8 \\ 2 \\ 5 \end{pmatrix}$.

(1) Write down the augmented matrix $\widetilde{A}$.

(2) Find the row echelon form of $\widetilde{A}$ through elementary row operations.

(3) Find $\operatorname{rank} A$ and $\operatorname{rank} \widetilde{A}$.

(4) Discuss how many solution(s) this system has.

[The exam questions continue on the next page.]

[Part III] Eigenvalues and Diagonalizaton (10.4 points)

Consider $A = \begin{pmatrix} 1 & 0 \\ 4 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & \mathrm{i} & -\mathrm{i} \\ -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}$.

(1) Describe the meaning of "a matrix M is diagonalizable."

(2) Find eigenvalues of A and corresponding eigenvectors.

(3) Diagonalize A.

(4) The matrix B has three different eigenvalues. One of them is -1. Find the other two.

(5) Find an eigenvector of B corresponding to the eigenvalue -1, and normalize it.

[Part IV] Properties of Matrices (5.6 points)

Prove the following statements.

(1) A unitary matrix is normal.

(2) An idempotent matrix is either singular or identity.

(3) If H is a $n \times n$ Hermitian matrix, $\vec{y}^{\dagger} H \vec{x}$ is the complex conjugate of $\vec{x}^{\dagger} H \vec{y}$ for any complex vectors $\vec{x}$ and $\vec{y}$ with n components.

[Part V] Extra Problem 1

Prove that eigenvalues of an Hermitian matrix are real.

[The exam questions continue on the next page.]

[Part VI] Extra Problem 2

> This extra problem is for students who have done the previous parts quickly and have some spare time. You **do not** get any points for this problem if your score for **[Part I]**–**[Part IV]** is less than 75%.

In physics, we often consider the exponential of a square matrix X. It is defined by

$$\exp X := I + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \cdots = \lim_{N\to\infty}\sum_{k=0}^{N}\frac{1}{k!}X^k. \tag{1}$$

As it is known that the limit always converges, $\exp X$ is always well-defined. For example,

$$\exp\begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix} = \begin{pmatrix} \exp d_1 & 0 & \cdots & 0 \\ 0 & \exp d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \exp d_n \end{pmatrix}, \tag{2}$$

$$\exp\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

and so on. Furthermore, it is known that

- $\exp X$ is always invertible. In particular, $\exp(-X)$ is the inverse matrix of $\exp X$.
- $\exp(\mathrm{i}Ht)$ is unitary for Hermitian H and real t.
- $\det(\exp X) = \exp(\operatorname{Tr} X)$, where $\operatorname{Tr} X = \sum_{i=1}^{n} X_{ii}$, i.e., the sum of the diagonal elements.

The proofs for the first two statements are easy, while the third statement is not trivial; its proof uses Jordan forms and is beyond this course. However, we are usually interested in $\exp(\mathrm{i}Ht)$, where H is Hermitian and t is real. In this case, the proof becomes easier since we can diagonalize $\mathrm{i}Ht$.

Here, we try to prove $\det(\exp X) = \exp(\operatorname{Tr} X)$ for a skew-Hermitian matrix X. You can use the above equations (1) and (2).

(1) Let $X = \mathrm{i}Ht$, where H is a $n \times n$ Hermitian matrix and t is real. Show X is skew-Hermitian.

(2) Since X is skew-Hermitian, there exists a matrix P such that $P^{-1}XP = D$, where the diagonal matrix D is given by $D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$ with λ_i being the eigenvalues of X. For a positive integer k, express X^k by P and D.

(3) Express $\exp(X)$ by P and D. Then, show $\det(\exp X) = \exp\left(\sum_{i=1}^{n}\lambda_i\right)$.

(4) Show $\sum_{i=1}^{n}\lambda_i = \operatorname{Tr} X$ to complete the proof. (You may assume, although unnecessary, that all the eigenvalues are different.)

[This is the end of the exam questions.]